1999 Mu Alpha Theta Tennessee Bowl
Alpha Division

Individual Questions

1. A function f from the integers to the integers is defined as follows: $f(n) = \begin{cases} n + 3, \text{ if n is odd} \\ \frac{n}{2}, \text{ if n is even} \end{cases}$

Suppose k is odd and $f(f(f(k))) = 125$. What is the sum of the digits of k?

Answer: 20
Solution: Since k is odd, the $f(k) = k + 3$; since $k + 3$ is even then $f(k + 3) = \frac{k+3}{2}$; If $\frac{k+3}{2}$ is even, then $f(\frac{k+3}{2}) = \frac{k+3}{4} = 125$; $k + 3 = 500$; $k = 497$ The sum of the digits $4 + 9 + 7 = 20$ If $\frac{k+3}{2}$ is odd, then $f(\frac{k+3}{2}) = \frac{k+3}{2} + 3 = 125$; $k = 45$. This value does not work
$f(f(f(45))) = f(f(48)) = f(24) = 12$

2. Find the two smallest positive values of x, in radians, if $0 \le x < 2\pi$ which makes the following true.
$\text{Csc}^2 x + \text{Sec}^2 x = 8$

Answer: $\frac{\pi}{8}, \frac{3\pi}{8}$

$\frac{1}{Sin^2 x} + \frac{1}{Cos^2 x} = 8$; $\frac{Cos^2 x + Sin^2 x}{Sin^2 x\, Cos^2 x} = 8$; $1 = 8\, Sin^2 x\, Cos^2 x$; $\frac{1}{8} = Sin^2 x\, Cos^2 x$

$\pm\sqrt{\frac{1}{8}} = Sin\, x\, Cos\, x$; $\pm\frac{\sqrt{2}}{2} = 2\, Sin\, x\, Cos\, x$; $\pm\frac{\sqrt{2}}{2} = Sin\, 2x$; $2x = \frac{\pi}{4}$; $x = \frac{\pi}{8}$; $2x = \frac{3\pi}{4}$ $x = \frac{3\pi}{8}$;

3. If $f(x) = ax^2 + bx + c$, find the ordered triple (a, b, c) of the quadratic function whose graph contains the points (1, 4), (- 1, 14), (0, 6).

Answer: (3, - 5, 6)
Solution: Starting with (0, 6) you will find that $c = 6$. Then using (1, 4), you get $4 = a + b + 6$ or $a + b = -2$
Using (- 1, 14) you get $14 = a - b + 6$ or $a - b = 8$. Solving the system you get $a = 3$, $b = -5$

4. Factor completely under the real numbers: $X^7 + X^5 - X^4 + X^3 - X^2 - 1$

Answer: $(X - 1)(X^2 + X + 1)^2(X^2 - X + 1)$

Solution: $(X^7 + X^5 + X^3) + (-X^4 - X^2 - 1)$

$X^3(X^4 + X^2 + 1) - 1(X^4 + X^2 + 1)$

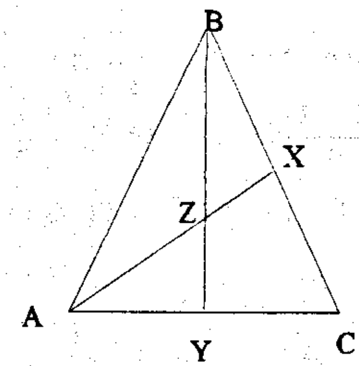
$(X^3 - 1)(X^4 + X^2 + 1)$

$(X - 1)(X^2 + X + 1)(X^2 - X + 1)(X^2 + X + 1)$

$(X - 1)(X^2 + X + 1)^2(X^2 - X + 1)$

Group Questions

5. If AC = 5.8 cm, AX = 5.1 cm, BY = 6.3 cm, X is the midpoint of segment BC and y is the midpoint of segment AC. Find the perimeter of triangle AYZ?



Answer: 8.4

Solution: $\overline{BY}$ and $\overline{AX}$ are medians. AX = 5.1

$AZ = \frac{2}{3}(5.1) = 3.4 \quad BY = 6.3 \Rightarrow ZY = \frac{1}{3}(6.3) = 2.1$ Y is the midpoint of $\overline{AC}$ Then $\overline{AY} = .5(5.8) = 2.9$ Perimeter = 2.9 + 2.1 + 3.4 = 8.4

6. An infinite geometric series has a sum of 8. If the sum of the first two terms is 2, find the first term(s).

Answer: $8 - 4\sqrt{3}$, $8 + 4\sqrt{3}$

Solution: In an infinite geometric series, $A_1 + A_2 + A_3 + \ldots$ $A_1 + A_2 = 2$; $A_1 + A_1 r = 2$;

$8 - 8r + r(8 - 8r) = 2$; $-8r^2 = -6$; $r = \pm\frac{\sqrt{3}}{2}$; $8 = \frac{A_1}{1 - \frac{\sqrt{3}}{2}}$; $A_1 = 8 - 4\sqrt{3}$;

$8 = \frac{A_1}{1 + \frac{\sqrt{3}}{2}}$; $A_1 = 8 + 4\sqrt{3}$

7. If n is a positive integer, D (n) is the sum of the squares of the positive divisors of n. Thus, D (1) = 1 and D (4) = $1^2 + 2^2 + 4^2 = 21$. What is the smallest positive integer n such that D (n) is at least 1999?

Answer: 40

Solution: Using the fact that D (mn) = D(m) D(n) when (m,n) = 1, we see that D(40) = D(8)D(5) = (85)(26)= 2210, while D(n) < 1999 for n < 40

8. A 12 member search team is to search the square area shown. Groups of three are selected at random to cover each of A, B, C, and D. Find the probability that Peter, Paul, and Mary of the group are selected and they search any area NOT D. Express answer as a fraction in simplest form.

A	B
C	D

Answer: $\frac{3}{220}$

Solution: There are $\frac{9}{12}$ appropriate places for Mary. The fact that Peter and Paul must follow would be $\frac{2}{11}$ and $\frac{1}{10}$, respectively. $\frac{9}{12} \cdot \frac{2}{11} \cdot \frac{1}{10} = \frac{3}{220}$

9A. Find the product of: $\sqrt{3(\sqrt[3]{2})} \cdot \sqrt{2(\sqrt[3]{3})} \cdot \sqrt[3]{2\sqrt{3}} \cdot \sqrt[6]{3}$

Answer: 6

Solution: $\sqrt{3(\sqrt[3]{2})} \cdot \sqrt{2(\sqrt[3]{3})} \cdot \sqrt[3]{2\sqrt{3}} \cdot \sqrt[6]{3} = 3^{\frac{1}{2}} \cdot 2^{\frac{1}{6}} \cdot 2^{\frac{1}{2}} \cdot 3^{\frac{1}{6}} \cdot 2^{\frac{1}{3}} \cdot 3^{\frac{1}{6}} \cdot 3^{\frac{1}{6}} = 2 \cdot 3 = 6$

9B. Simplify the expression: $\sqrt[n]{\frac{4^{2n+1} + 2^{4n+1}}{6}}$

Answer: 16

Solution: $\sqrt[n]{\frac{4^{2n+1} + 2^{4n+1}}{6}} = \sqrt[n]{\frac{2^{4n+2} + 2^{4n+1}}{6}} = \sqrt[n]{\frac{2^{4n}(2^2+2)}{6}} = \sqrt[n]{2^{4n}} = 16$

10A. Find the solution set in the form of a + bi of the following equation.

$$ix^2 - (3 + 6i)x - 1 + 9i = 0$$

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Answer: 5 - 4i ; 1 + i

Solution: Using the quadratic formula:

$$x = \frac{3 + 6i \pm \sqrt{(3 + 6i)^2 - 4(i)(-1 + 9i)}}{2i} = \frac{3 + 6i \pm \sqrt{9 + 40i}}{2i} =$$

$$\frac{3 + 6i \pm (5 + 4i)}{2i} \rightarrow \frac{8 + 10i}{2i} = 5 - 4i \quad \text{Also} \quad \frac{-2 + 2i}{2i} = 1 + i$$

10B. Given log 2 = a, log 3 = b, and log 7 = c, find log (10!) in terms of a, b, and c.

Answer: 6a + 4b + c + 2

Solution: $\log(10!) = \log(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) =$

$$\log 10 + 2\log 3 + 3\log 2 + \log 7 + (\log 2 + \log 3) + (\log 10 - \log 2) + 2\log 2 + \log 3 + \log 2 + \log 1 =$$

6a + 4b + c + 2

11A. Find the value of Cos 2x - Sin (90 ° + x) if Tan x = $-\frac{3}{4}$ and Sin x > 0.

Answer: $\frac{27}{25}$

Solution: Since Tan x = $-\frac{3}{4}$ and Sin x > 0 , then Cos x = $-\frac{4}{5}$ and Sin x = $\frac{3}{5}$ Cos 2x = $\cos^2 x - \sin^2 x$

$$(-\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}; \quad \sin(90 + x) = \sin 90° \bullet \cos x + \cos 90° \bullet \sin x = -\frac{4}{5}$$

$$\frac{7}{25} + \frac{4}{5} = \frac{27}{25}$$

11B. Find the value of $\dfrac{Tan\,140° \; Tan\,70°}{1 - Tan\,140° \cdot Tan\,70°}$

Answer: $\dfrac{\sqrt{3}}{3}$

Solution: $\dfrac{Tan\,140° \; Tan\,70°}{1 - Tan\,140° \cdot Tan\,70°} = \text{Tan}\,(140° + 70°) = \text{Tan}\,210° = \dfrac{\sqrt{3}}{3}$

11C. Find Arccos (Sin $\dfrac{5\pi}{4}$) in radians.

Answer: $\dfrac{3\pi}{4}$

Solution: $\text{Sin}\dfrac{5\pi}{4} = -\dfrac{1}{\sqrt{2}}$ $\quad$ $\text{Arccos}\left(-\dfrac{1}{\sqrt{2}}\right) = \dfrac{3\pi}{4}$

11D. Find the value of $\text{Cos}\left[2\,\text{Arcsin}\left(-\dfrac{3}{5}\right)\right]$

Answer: $\dfrac{7}{25}$

Solution: $\text{Cos}\,(2x) = 1 - 2\,\text{Sin}^2 x = 1 - 2\left(-\dfrac{3}{5}\right)^2 = 1 - 2\left(\dfrac{9}{25}\right) = \dfrac{7}{25}$

12A. Simplify:

$$\left[4^{\frac{-3}{2}} \cdot \sqrt{2^{12}} + \left(\frac{1}{8}\right)^{\frac{-2}{3}}\right] \div (27)^{\frac{1}{3}}$$

Answer: 4

Solution: $\left[4^{\frac{-3}{2}} \cdot \sqrt{2^{12}} + \left(\frac{1}{8}\right)^{\frac{-2}{3}}\right] \div (27)^{\frac{1}{3}} = \left[\frac{1}{8} \cdot 2^6 + 4\right] \div 3 = \left[2^{-3} \cdot 2^6 + 4\right] \div 3 = \left[2^3 + 4\right] \div 3 = 4$

12B. Solve: $2\sqrt{x} + 2x^{\frac{-1}{2}} = 5$

Answer: $\frac{1}{4}, 4$

Solution: $2\sqrt{x} + 2x^{\frac{-1}{2}} = 5 \qquad 2\sqrt{x} + \frac{2}{\sqrt{x}} - 5 = 0 \qquad 2x + 2 - 5\sqrt{x} = 0$

$$(2\sqrt{x} - 1)(\sqrt{x} - 2) = 0 \quad x = \frac{1}{4} \quad x = 4$$

12C. When the following expression is simplified via rationalizing the denominator, the denominator becomes what?

$$\frac{3}{2(\sqrt[3]{3}) + \sqrt[3]{5}}$$

Answer: 29

Solution: $(2(\sqrt[3]{3}) + \sqrt[3]{5})(4(\sqrt[3]{9}) - 2(\sqrt[3]{15}) + \sqrt[3]{25}) =$

$8(\sqrt[3]{27}) - 4(\sqrt[3]{45}) + 2(\sqrt[3]{75}) + 4(\sqrt[3]{45}) - 2(\sqrt[3]{75}) + 5 \quad 29$

12D. If $f(1) = i$, $f(2) = 3i$, $f(n+1) = 2f(n) - f(n-1)$ for $n \geq 2$, find $f(1999)$.

Answer: 3997 i

Solution: $f(1) = i$, $f(2) = 3i$, $f(3) = 5i$, $f(4) = 7i$, $f(n) = (2n-1)i$ Therefore, $f(1999) = 3997i$