1992 National Mu Alpha Theta Convention

Functions Topic Test Answers:

1. A	16. C
2. C	17. B
3. D	18. B
4. C	19. B
5. C	20. A
6. B	21. D
7. A	22. A
8. A	23. D
9. D	24. D
10. C	25. C
11. D	26. C
12. B	27. C
13. B	28. B
14. C	29. C
15. D	30. D

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1992 MAΘ Convention
FUNCTIONS TEST
Solutions

A 1. We write

$$8 = f(x) = x^{4/3} - 8$$

so that $x^{4/3} = 16$,

or $x = \boxed{8}$.

C 2. By definition of inverse, $\boxed{f(g(x)) = x}$.

D 3. A function takes any number to a unique number; thus $\sin^{-1}(x)$, which takes a number to a set of numbers (for example, $\sin^{-1}(1) = \dots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \dots$) is not a function. However, $|x|$, $[x]$, and $\log x$ all have unique values given x. The answer is $\boxed{\text{three}}$.

4. The domain of $\sin x$ is all reals, so there is no restriction on $|\sqrt{\log x + 3}|$; the domain of $|x|$ is all reals, so there is no restriction on $\sqrt{\log x + 3}$. However, something under a square root must be zero or positive, so

$$\log x + 3 \geq 0$$
$$\log x \geq -3$$
$$\boxed{x \geq e^{-3}}.$$

This domain is also in the domain of $\log x$, so there is no further restriction.

C 5. $f(17) = f\left(\frac{17}{2} \cdot 2\right) = \frac{17}{2} f(2) = \frac{17}{2}(5) = \boxed{\frac{85}{2}}$.

B 6. $$f(4x) = \frac{4x}{4x-1} = \frac{4x/(x-1)}{(4x-1)/(x-1)} = \frac{4\frac{x}{x-1}}{4\frac{x}{x-1} - \frac{1}{x-1}} = \frac{4f(x)}{3\frac{x}{x-1} + \frac{x-1}{x-1}}$$

$$= \boxed{\frac{4f(x)}{3f(x)+1}}$$

A 7. Call the roots $a-r$, a, $a+r$. From the relationship between roots and coefficients we know that b is the product of the roots:

$$b = a(a-r)(a+r) = a(a^2-r^2).$$

We also have $12 = a+(a-r)+(a+r) = 3a$

and $37 = a(a-r)+a(a+r)+(a-r)(a+r)$

$$= 3a^2 - r^2.$$

Thus $a=4$, $a^2=16$, and $r^2=11$, so that $b = \boxed{20}$.

A 8. We must have

$$a+x > 0 \quad \underline{\text{AND}} \quad a-x>0$$

$$x > -a \quad \underline{\text{AND}} \quad x<a$$

so the answer is $\boxed{-a<x<a}$.

D 9. $f(x)=\frac{1}{x}$ approaches $-\infty$ as $x \to 0$ from the left and $+\infty$ from the right, so it has a discontinuity at 0.

$f(x)=|x|$ is continuous.

$f(x)=[x]\sin \pi x$ has possible discontinuities at the integers. If we come into integer n from the left or right, however, $\sin \pi x \to 0$, so that $f(x) \to 0$ from both sides, and the function is continuous.

$f(x)=e^{|x|}$ is continuous, since it is the composition of e^x and $|x|$, two continuous functions.

The answer is $\boxed{\text{three}}$.

C 10. $\ln(4x^4-8x^3-17x^2+40x-14)=0 \Rightarrow 4x^2-8x^3-17x^2+40x-14=1$
$\Rightarrow 4x^2-8x^3-17x^2+40x-15=0$. From the rational root theorem, possible rational roots are restricted to $\pm5, \pm3, \pm1$, $\pm\frac{5}{2}, \pm\frac{5}{4}, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{2}, \pm\frac{1}{4}$. Trying the easy ones first we find $\frac{1}{2}$ and $\frac{3}{2}$ work, and the equation left over is $4x^2-20=4(x^2-5)=0$, which has only roots $\pm\sqrt{5}$.
The sum of the rational roots is $\frac{1}{2}+\frac{3}{2}=\boxed{2}$.

D 11. $f(x) = f(a \cdot \frac{x}{a}) = \log_a \frac{x}{a} = \log_a x - \log_a a = \boxed{\log_a x - 1}$.

B 12. Write $y = \frac{ax+b}{cx+d} \Rightarrow cxy + dy = ax + b$

$$\Rightarrow ax - cxy = dy - b$$

$$\Rightarrow x = \frac{dy-b}{-cy+a}$$

so $f^{-1}(x) = \frac{dx-b}{-cx+a}$.

B 13. Recall that a polynomial of degree n has n roots, real and nonreal. Also recall that nonreal roots must come in conjugate pairs. The polynomial in the problem has an odd number of roots, and so has an even number of nonreal roots and an odd number of real roots. The only answer of the answers given is $\boxed{\text{an odd number of real roots.}}$

C 14. e^x has the inverse $\ln x$, which is well-defined, for $x > 0$. If we substitute $-x$ for $+x$ in $\frac{e^x + e^{-x}}{2}$, we get the same thing back, so this function is not 1:1 (two different values, x and $-x$, go to the same value) and cannot have a well-defined inverse function.

$\frac{e^x - e^{-x}}{2}$ is increasing, that is, if $a > b$, $\frac{e^a - e^{-a}}{2} > \frac{e^b - e^{-b}}{2}$, since $e^a > e^b$ and $e^{-a} < e^{-b}$. Thus it is 1:1 and has a well-defined inverse.

$e^{x^2} = e^{(-x)^2} \Rightarrow$ not 1:1 $\Rightarrow$ no inverse.

The answer is $\boxed{\text{two.}}$

D 15. Relation between roots and coefficients: $r_1 + r_2 + r_3 = -6$, $r_1 r_2 + r_1 r_3 + r_2 r_3 = -12$.

$(r_1^2 + r_2^2 + r_3^2) = (r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_1 r_3 + r_2 r_3) = 36 + 24 = \boxed{60}$.

C 16. $f(1)=1,\ f(2)=1,\ f(3)=2,\ f(4)=4,\ f(5)=8, \ldots$

$f(n)=\begin{cases}1, & n=1\\ 2^{n-2}, & n\neq 1\end{cases}$. Thus the range is $\{1, 2, \ldots, 2^{15}\}$

and the sum is $1+2+\cdots+2^{15} = \boxed{2^{16}-1}$.

B 17. $\log ax = \log a + \log x \neq a \log x$

$\log \frac{x}{y} = \log x - \log y$

$\log x^a = a \log x \neq (\log x)^{\log a}$

$\log^{-1} x = e^x$; $e^1 = e \neq 0$ $\Rightarrow$ The answer is $\boxed{\text{one}}$.

B 18. Change n into its prime decomposition: $f(n) = f(p_1 p_2 p_3 \cdots p_k)$

$= f(p_1) f(p_2) \cdots f(p_k) = p_1^{1/2} p_2^{1/2} \cdots p_k^{1/2} = n^{1/2} \Rightarrow f(n) = \sqrt{n}$.

Thus, i. true ii. false iii. false iv. false. The answer is $\boxed{\text{one}}$.

B 19. Perform the compositions: $a_1(a_2x+b_2)+b_1 = a_1a_2x + (a_1b_2+b_1) \Rightarrow$ yes

$a_1(a_2x^2+b_2x+c_2)^2+\cdots = a_1a_2^2x^4+\cdots \Rightarrow$ no

$a_n(b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0)^n + a_{n-1}(b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0)+\cdots+a_0$

is a polynomial $\Rightarrow$ yes

Given p, q polynomials, $p(12)=q(12)=0$, $p(q(12))=0$ is only

certainly true if $q(12)=12$ or $p(0)=0 \Rightarrow$ not always

The answer is $\boxed{\text{one}}$.

A 20. $f(x,y,a) = x^2-y^2-a=0 \Rightarrow x^2-y^2=a$ if $|y|>|x|$, $\boxed{a<0}$.

D 21. $\sin 10\pi x$ is between -1 and $1 \Rightarrow$ no ; $5\sin 2\pi x$ goes from -5 to $5 \Rightarrow$ yes ;

$10(0)-5=-5$ and $10(1)-5=5 \Rightarrow$ yes; $6(0)^2+4(0)-5=-5$ and $6(1)^2+4(1)-5=5$

The answer is $\boxed{\text{three.}}$

A 22. $F(\frac{25}{4}, \frac{16}{3}) = F(\frac{21}{4}, \frac{16}{3})+1 = \cdots = F(\frac{1}{4}, \frac{16}{3})+6 = F(\frac{1}{4}, \frac{13}{3})+7$

$= \cdots = F(\frac{1}{4}, \frac{1}{3})+11 = -\frac{1}{12}+11 = \boxed{\frac{131}{12}}$

D 23. We use the reversal trick to simplify:

$$f(1-x)+(1-x)f(x)=5$$

$$f(1-(1-x))+(1-(1-x))f(1-x)=5$$

$$f(x)+x\,f(1-x)=5$$

$$x(1-x)f(x)+x\,f(1-x)=5x$$

Subtracting, $(x(1-x)-1)f(x)=5x-5$

or $f(x)=\dfrac{5x-5}{-x^2+x-1}$. [Note $-x^2+x-1$ has no real zeros so this division is legal.]

Thus $f(5)=\boxed{\dfrac{-20}{21}}$.

D 24. $f(g(x))=a e^{bc\ln dx}=a e^{\ln (dx)^{bc}}=a d^{bc}x^{bc}$

$g(f(x))=c\ln d a e^{bx}=c(\ln da+\ln e^{bx})=c(\ln da+bx)$

We must have $ad^{bc}x^{bc}=c\ln da+bcx$ for all x. Thus $bc=1$, since x-exponents must be equal. Also $c\ln da=0$, since it is an unmatched constant. Since $c\neq 0$, $\ln da=0$, and $da=1$. Letting $bc=ad=1$ gives $adx=bcx$, or $x=x$. This is a necessary & sufficient condition.

C 25. i. $|x|+|y|\geq 0$; $|x|+|y|=0$ iff $x=y=0$; $|x|+|y|=|x|+|0|+|y|+|0|$ ⇒ yes

ii. for negative x and y this is negative. ⇒ no

iii $e^x\geq 0$; $e^x\neq 0$ if $x=y=0$ ⇒ no

iv. $(x^2+y^4)^{1/2}\geq 0$, $=0$ iff $x=y=0$. $(x^2+y^4)^{1/2}\leq x+y^2$ ⇒ yes

The answer is $\boxed{\text{two}}$.

C 26. Take the logs of both sides to get

$$\log f_b(x)=a\log 2+\log f_b(x-a),$$

or, naming $\log f_b(x)$ $F_b(x)$,

$$F_b(x)=a\log 2+F_b(x-a).$$

Thus $F(x)$ is linear: $F_b(x)=x\log 2+F_b(0)\Rightarrow \log f_b(x)=x\log 2+\log b$, or $f_b(x)=b2^x$, so that

$$f(2x)=c2^{2x}=\boxed{\frac{c}{b^2}[f_b(x)]^2}$$

C 27. A function has an inverse function if and only if it is one-to-one. Thus a function on A must be a permutation of A if it is to be invertible; there are $13!$ permutations of A. The total number of functions from $A \to A$ can be counted by observing that each of 13 elements can go to any of 13 elements, for 13^{13} functions. The fraction is $\frac{13!}{13^{13}} = \boxed{\frac{12!}{13^{12}}}$.

B 28. We use the reversal trick — twice.

$$f(x) + 2f\left(\tfrac{1}{1-x}\right) = x$$
$$f\left(\tfrac{1}{1-x}\right) + 2f\left(1-\tfrac{1}{x}\right) = \tfrac{1}{1-x} \rightarrow 2f\left(\tfrac{1}{1-x}\right) + 4f\left(1-\tfrac{1}{x}\right) = 2\tfrac{1}{1-x}$$
$$f\left(1-\tfrac{1}{x}\right) + 2f(x) = 1-\tfrac{1}{x} \rightarrow 4f\left(1-\tfrac{1}{x}\right) + 8f(x) = 4-\tfrac{4}{x}.$$

Subtracting and adding to cancel the extraneous, we have

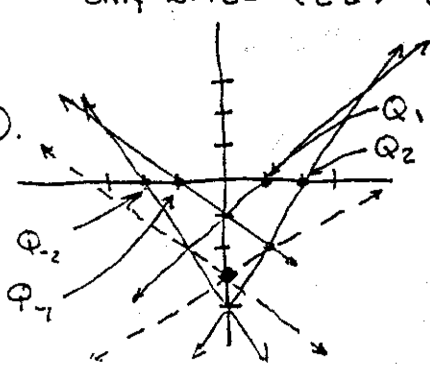
$$9f(x) = 4 - \tfrac{4}{x} - 2\tfrac{1}{1-x} + x,$$

so $9f(2) = 4 - 2 + 2 + 2 = 6 \Rightarrow f(2) = \boxed{2/3}$.

C 29. This is the sum of a translation, $\binom{x}{y} \to \binom{x}{y} + \binom{e}{f}$, and a linear transformation $\binom{x}{y} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y}$. The translation shifts everything bey the same amount, so clearly it preserves distance. The linear transformation must then preserve distance. Since $0 \to 0$, $\binom{1}{0}$ must go to a point on the unit circle. $\binom{1}{0} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{1}{0} \to \binom{a}{c}$, so $a^2+c^2=1$. Similarly, $\binom{0}{1} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{0}{1} \to \binom{b}{d}$, so $b^2+d^2=1$. Also the distance between $\binom{1}{0}$ and $\binom{0}{1} = \sqrt{2}$ must stay the same, so $(a-b)^2+(c-d)^2 = a^2-2ab+b^2+c^2-2cd+d^2 = 2-2(ab+cd) = 2$, or $ab+cd=0$. To show these restrictions sufficient we need only write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \pm\sin\theta \\ \sin\theta, & \pm\cos\theta \end{pmatrix}$, so it is a reflection or rotation.

Note $\binom{x}{y}$, the column vector, is the same as a point (x, y).

D 30.



This is simple once you get past notation.

$G(Q_2) = \{y = 2x-4\}$, $G(Q_{-1}) = \{y = -x-1\}$,
$G(Q_{-2}) = \{y = -2x-4\}$, $G(Q_1) = \{y = x-1\} \Rightarrow$
$H(G(Q_2), G(Q_{-1})) = (1,-2) \Rightarrow G(H(G(Q_2), G(Q_{-1})) = \{y = x -$
$H(G(Q_{-2}), G(Q_1)) = (-1,-2) \Rightarrow$ —— Q_{-2} —— $Q_1 = \{y = -x -$
$\Rightarrow$ intersection at $\boxed{(0,-3)}$